linear equations:

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) + I_R\omega_2\Omega - I_m\ddot{\theta} = 0 \quad (7)$$

$$I_2\dot{\omega}_2 - \omega_1\omega_3(I_3 - I_1) - I_R\omega_1\Omega - I_m\omega_3\dot{\theta} = 0 \quad (8)$$

$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) + I_R\dot{\Omega} + I_m\omega_2\dot{\theta} = 0 \quad (9)$$

These equations involve five unknowns, viz., ω_1, ω_2, ω_3, θ, and Ω and, therefore, two additional equations are required to form a complete set. Clearly, one is provided by the equation of rotor motion, given by

$$I_R(\dot{\omega}_3 + \dot{\Omega}) = T \quad (10)$$

where T represents the magnitude of the moment applied about the rotor axis, while the other equation is obtained by considering the rotational motion of the damping system caused by the inertial forces. Thus, by computing the moments about the a_1 axis due to both the a_2 and a_3 components of inertial forces acting on all the four masses of the nutation damper considered, and noting that the resulting torque simply equals the sum of the damping torque and the restoring torque offered by the torsional spring arrangement about the point 0, this latter equation may be found as

$$I_m\ddot{\theta} - I_m\dot{\omega}_1 + C\dot{\theta} + K\theta = 0 \quad (11)$$

where C denotes the damping rate constant and K is the spring constant of the proposed damper.

Stability Analysis

The Eqs. (7–11) describe completely the behavior of the dual-spin system of Fig. 1 with the proposed four-mass nutation damper. It may be seen that all these equations are trivially satisfied for the solution

$$\omega_3 = \text{const}$$
$$\Omega = \text{const}$$

and

$$\omega_1 = \omega_2 = \theta = 0 \quad (12)$$

provided that $T = 0$, meaning thereby that the motor torque balances the bearing friction. Stability of this solution can be studied by considering the following set of linear equations:

$$\dot{\omega}_1 + \lambda\omega_2 - I_{m_1}\ddot{\theta} = 0 \quad (13)$$

$$\dot{\omega}_2 - \lambda\omega_1 - I_{m_1}\omega_3\dot{\theta} = 0 \quad (14)$$

and

$$-\dot{\omega}_1 + \ddot{\theta} + \beta\dot{\theta} + p^2\theta = 0 \quad (15)$$

where the variables now represent small increments around the stable solution given by Eq. (12) and where the constants are defined as follows:

$$\lambda = [(I_3 - I_T)\omega_3 + I_R\Omega]/I_T \quad (16)$$

$$I_{m_1} = I_m/I_T \quad (17)$$

where, due to the symmetry, it is assumed that $I_1 = I_2 = I_T$, I_T being the transverse moment of inertia of the total system and, additionally, $\beta = C/I_m$ and $p^2 = K/I_m$.

By using Eqs. (13–15), the characteristic equation may be obtained as

$$(1 - I_{m1})s^4 + \beta s^3 + (p^2 + \lambda^2 + I_{m_1}\lambda\omega_3)s^2 + \lambda^2\beta s + \lambda^2 p^2 = 0 \quad (18)$$

This is a fourth-order polynomial with constant coefficients. The application of Routh-Hurwitz criterion to this polynomial will give both the necessary and sufficient conditions for asymptotic stability of the solution expressed by Eq. (12). Thus, for stability, the procedure yields

a) $(1 - I_{m_1}) > 0$

b) $\beta > 0$

c) $(p^2 + I_{m_1}\lambda\omega_3 + \lambda^2 I_{m1}) > 0$

d) $I_m\beta\lambda^3(\omega_3 + \lambda) > 0$

and

e) $\lambda^2 p^2 > 0$ (19)

Note that the stability conditions given do not involve the kind of design constraints which restricted the performance of the other available nutation dampers. From Eq. (19), the following stability criteria may finally be derived:

a) $\lambda > 0$

b) $\beta > 0$ (20)

Use of a Circular Disk or Wheel

A logical extension of the four-mass nutation damper described will be the case where a circular disk or wheel of uniform mass distribution is used to experience the torque produced by the inertial forces. It can be shown that the same stability criteria given by Eq. (20), will also hold in this situation with the only difference that I_m will now represent the moment of inertia of the wheel used.

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Comparison of Beam Impact Models

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Nomenclature

A = cross-sectional area
A_s = cross-sectional area contributing to resistance to shear
E = modulus of elasticity
G = modulus of rigidity
I = cross-sectional moment of inertia
L = span length
s = spring stiffness
t = time
x = space variable measured along beam axis
y = vertical deflection of neutral axis
ψ = rotation of cross section about neutral axis

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Introduction

THE determination of a valid and tractable mathematical model is of primary importance when dealing with physical problems. Often, in the case of beam impact, this determination must be made between the elementary Euler-Bernoulli and the Timoshenko theories.

Assuming the validity of a one-dimensional model, the principal liability of the Euler-Bernoulli theory is that the resulting partial differential equation is not hyperbolic and therefore does not correctly predict the propagation of discontinuities which are inherent to an impact problem.[1] The simplest one-dimensional model of beam behavior which leads to a hyperbolic partial differential equation is the Timoshenko beam which, in addition to the effects considered in the elementary theory, takes into account deformation due to transverse shear and the rotary inertia of beam elements.[2]

The following quantitative result sheds some light on the adequacy of the elementary theory, as a function of an impact parameter, to predict beam response. The case of a finite beam resting on spring supports is considered. The transverse impact, suggested by the importance of beam rattling when subjected to a sonic boom, is imposed in the form of an initial transverse uniform velocity. In addition to the application to sonic boom problems, the results may be applicable to other problems such as packaging.

Equations

1) The elementary Euler-Bernoulli model of transverse bending results in the partial differential equation

$$EIy_{xxxx} + \rho A y_{tt} = 0$$

while the initial and boundary conditions equivalent to a finite spring supported span subjected to a uniform velocity input are

$$\begin{array}{ll} \text{for } t = 0 & y = 0;\, y_t = 1 \\ \text{for } x = 0 & y_{xx} = 0;\, EIy_{xxx} - sy = 0 \\ \text{for } x = L & y_{xx} = 0;\, EIy_{xxx} + sy = 0 \end{array}$$

This formulation has been solved using classical eigenfunction expansion techniques.[3] The solutions are obtained in series form with the internal shear and moment given by

$$V = EIy_{xxx},\ M = -EIy_{xx}$$

2) The equations of motion of a Timoshenko beam may be written as

$$\begin{gathered} EIy_{xx} + A_sG(y_x - \psi) - \rho I \psi_{tt} = 0 \\ \rho A y_{tt} - A_sG(y_{xx} - \psi_x) = 0 \end{gathered}$$

where $M = EI\psi_x$ and $V = A_sG(y_x - \psi)$.

The corresponding initial and boundary conditions are given by

$$\begin{array}{ll} \text{for } t = 0 & y = 0;\, y_t = 1;\, \psi = 0;\, \psi_t = 0 \\ \text{for } x = 0 & \psi_x = 0;\, A_sG(y_x - \psi) - sy = 0 \\ \text{for } x = L & \psi_x = 0;\, A_sG(y_x - \psi) + sy = 0 \end{array}$$

The Timoshenko equations may be analyzed using a dual eigenfunction expansion.[4] The problem can be shown to be self adjoint, resulting in real eigenvalues and a complete orthogonal set of eigenfunctions. A computer subroutine, using the method of interval halving, was required to solve for the roots of the characteristic equation governing the eigenvalues.

The previous sets of equations may be expressed in nondimensional form yielding three parameters. The slenderness ratio (L/r), where L is the span length and r the radius of gyration of the cross section, is strictly a function of the beam geometry. The ratio of the velocity of propagation of a dilatational wave to that of a shear wave (c_l/c_s) depends on both beam geometry and material properties. The third parameter, $K \equiv sL/(A_sG)$ where s is the spring stiffness, is a measure of the relative stiffness of the spring to that of the beam.

Results

A comparison between the two theories is shown in Figs. 1 and 2. The graphs show the variation of moment at midspan and shear at end span, as a function of time. The results are presented in dimensionless form, the abscissa being c_st/L, i.e., the ratio of time to the time required for a shear wave to travel the span.

A representative value of the slenderness ratio $(L/r = 40)$ was chosen and two values of the spring stiffness parameter $(K \equiv sL/A_sG)$ are presented. $K = 10^2$ represents a "soft" support while $K = 10^4$ constitutes a "stiff" spring, approaching a rigid support condition. In all calculations the ratio c_l/c_s was assumed equal to 1.47, a value corresponding to a rectangular cross section and a value of Poisson's ratio equal to 0.3.

Immediately apparent in Fig. 1 is the inability of the elementary Euler-Bernoulli theory to correctly predict the shear response, and therefore shear stresses, even in the case of a relatively soft spring support. The discrepancies are seen to be most pronounced in the vicinity of high oscillations, i.e., integer values of c_s/tL. These regions represent reflected wave fronts as predicted by the hyperbolic Timoshenko equations. These results, while not surprising for the case of high impact, i.e., large K, reveal that the inadequacy of the elementary theory in this regard extends even to the case of a soft spring representing a "moderate" impact.

Somewhat reassuring are the results of Fig. 2 showing, in general, reasonable agreement between the predicted values of the moment at midspan. This result would tend to vali-

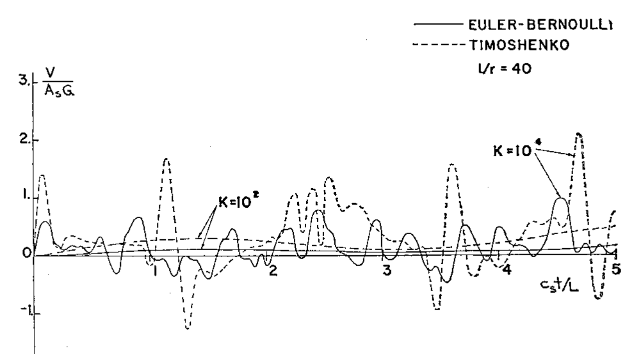

Fig. 1 Variation of shear at end span; various spring constants. Comparison of Euler-Bernoulli and Timoshenko theories.

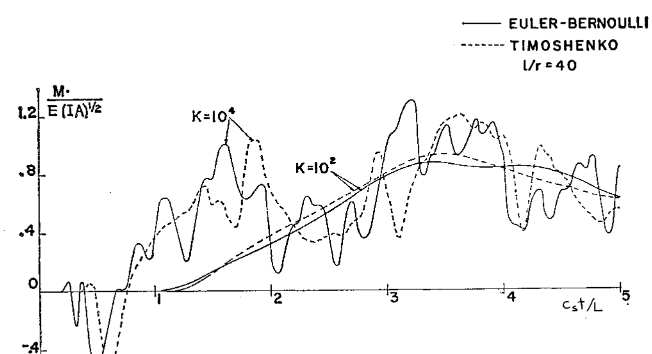

Fig. 2 Variation of moment at midspan; various spring constants. Comparison of Euler-Bernoulli and Timoshenko theories.

date the use of the Euler-Bernoulli theory in determining flexural stresses, even in the case of reasonably high impact.

It might also be of interest to note that in general, the dependence of the response on the slenderness ratio was one of increased oscillatory motion with higher L/r.

An extension of these results to the limiting cases of rigid supports and large slenderness ratios presented some mathematical problems. The difficulties, considering the Timoshenko model, involve poor convergence of the series solutions and rapid oscillations in the vicinity of the wave fronts. However, the method of singular perturbation and the use of the theory of characteristics can be used to relieve the problem. Results of this analysis will be represented in a future publication.

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Steady-State Burning of Double-Base Propellants at Low Pressures

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Nomenclature

A = cross-sectional area of propellant
c = heat capacity of propellant
E = energy transmitted to solid
k = thermal conductivity of propellant
r = linear burning rate
T = temperature
y = coordinate parallel to the axis of propellant; the burning surface at $y = 0$
ρ = mass density of propellant
0 = initial state
p = propellant
s = surface

Introduction

ONE of the difficulties in constructing a mathematical model to describe the deflagration characteristics of double-base propellants is the lack of understanding of the role of each parameter. There has not been a sufficient number of critical experiments on double-base propellants to characterize its combustion mechanism. The extent of the so-called solid phase (or condensed phase) reactions and the surface reaction, the mechanism of decomposition, the magnitudes and effects of heat transfer from the gas phase to the solid phase, the influence of the chemical reaction at the surface on the burning rate, and the mechanism of the gas-phase reaction are not clarified yet. This Note deals with the burning rate of M-2 double-base propellant at various initial temperatures and at low pressures ranging from 2.4 to 165 psia.

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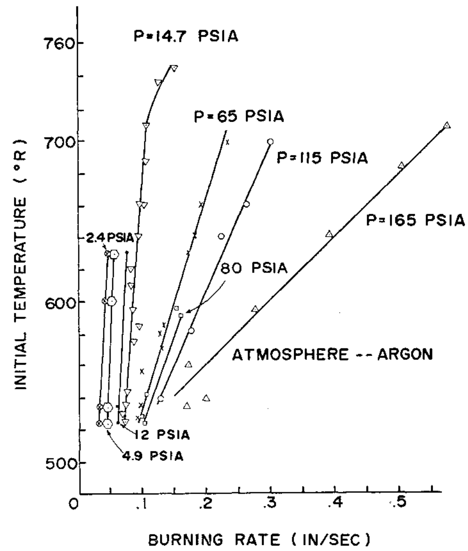

Fig. 1 Experimental results...burning rate vs initial temperature at various pressures (in argon).

The deflagration mechanism of double-base propellants has been investigated by many researchers in the past. Heller and Gordon,[1] Rice and Ginelli,[2] and Parr and Crawford[3] investigated the reactions and existence of "foam," "fizz," "dark," and "flame" zones. Wilfong et al.[4] also investigated the burning rate as a function of the chamber pressure and initial temperature. Recently there have been many publications coming out of Russian[5,6] research on double-base propellants.

Experimental Apparatus and Procedure

The experimental apparatus was arranged to accommodate ignition of the propellant in a gaseous atmosphere at low- and high-ambient pressures and low- and high-initial propellant temperatures. The combustion chamber was made of stainless steel. The propellant was placed vertically in the chamber and was ignited at the lower end by a heater. The burning rate was determined using a fuse wire technique. Leads were connected through the chamber so that the burning ing rate and temperature of the propellant could be recorded. A quartz window was placed in one end of the chamber in order to observe the burning process.

The chamber was filled with either argon or air during the experiment. The gases entering the chamber were either heated or cooled before entering the chamber in order to raise or lower the initial propellant temperature. The initial temperature of the propellant was measured by placing an iron-constantan thermocouple through a hole in the propellant. The physical properties of M-2 double-base propellant are given by Suh et al.[7] The propellant was coated in order to insure a cigarette-type burning. The coated propellant was dried for 5 hr at 160°F.

Experimental Results

Burning rate measurements

Figure 1 shows the plot of the burning rate vs the initial temperature of the propellant obtained in argon at pressures ranging from 2.4 to 165 psia. It should be noted that the curves of the experimental results obtained at 14.7 psia or higher tend to cross each other at about 520°R. The extrapolation of these curves indicates that when the initial tem-